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A CONTRIBUTION TO THE THEORY OF VOLTAGE GRAPHS

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Voltage graphs, one of the main tools for constructing graph embeddings, appear to be useful in various areas of graph theory and combinatorics.

The present paper concerns several problems in voltage graph theory such as equivalence and regularity of coverings generated by (permutation) voltage graphs, automorphism groups and some other things.

Introduction

Voltage graphs are introduced by Gross [3] and generalized to permutation voltage graphs by Gross and Tucker [5]. These two devices provide a method of constructing covering spaces (namely graphs) over a given base graph.

In the usual treatment of covering spaces, a conjugacy class of subgroups of the fundamental group of the base space is associated to a covering. In 1977 Ezell [2] observed that one can well represent an n-fold covering by a class of homomorphisms from the fundamental group of the base space to the symmetric group S_n. A deep connection between coverings and permutations was known long ago and the method introduced in [2] is in part implicitly contained in the textbook on topology by Seifert and Threlfall [9] and in Reidemeister's book [8].

We apply this view in Section 2 to obtain a necessary and sufficient condition for two voltage assignments to generate topologically equivalent coverings. This question is of interest in topological graph theory since lifting of topologically equivalent embeddings by topologically equivalent coverings yields equivalent embeddings.

In Section 3, automorphism groups of coverings generated by voltage assignments are under study. We obtain a formula for these groups which is analogous to a well known one employing fundamental groups, (cf. [7]). This enables us to compute automorphism groups using certain subgroups of voltage groups, the latter ones being finite, as usual. Automorphism groups of Schreier coset graphs and Cayley colour graphs can also be determined by means of this formula.

While coverings generated by ordinary voltage graphs are always regular, permutation voltage graphs generate regular as well as irregular coverings. In the last section of this paper we give a criterion of regularity of coverings generated by permutation voltage assignments. The proof of this theorem takes advantage of a new characterization (see Section 1) of the image of the fundamental group by the homomorphism to S_n or to the voltage group. Such a characterization for ordinary voltage graphs has first been given in [1].

1. Voltage graphs and characteristic groups

A graph in this paper is always a finite oriented 1-dimensional CW-complex. Thus, both multiple- and self-adjacencies are possible.

For topological notions such as covering spaces and transitive group actions we refer to [7].

Given a connected graph K, an ordinary voltage assignment (OVA) $\varkappa$ on K in the (usually finite) group G is a mapping assigning to each direction e^+ and e^- of each edge e an element of G such that $\varkappa(e^-) = \varkappa(e^+)^{-1}$. We often write $\varkappa_e$ instead of $\varkappa(e)$. A permutation voltage assignment (PVA) is defined in a similar way, but we use a symmetric group S_n instead of G. The pair $(K, \varkappa)$ is called an ordinary or a permutation voltage graph according to the type of the assignment used. The main difference between OVA's and PVA's is in the way the derived graph $K^\varkappa$ is constructed. In the OVA case $V(K^\varkappa) = V(K) \times G$ and $E(K^\varkappa) = E(K) \times G$. If e is an edge from u to v in K, then (e, g) is an edge from (u, g) to $(v, g \cdot \varkappa_e)$. In the PVA case $V(K^\varkappa) = V(K) \times \{1, 2, \ldots, n\}$ and $E(K^\varkappa) = E(K) \times \{1, 2, \ldots, n\}$. Now, if e is from u to v in K, then (e, i) is from (u, i) to $(v, (i)\varkappa_e)$. The natural projection $p : K^\varkappa \to K$ which erases the second coordinate is a topological covering. If $\varkappa$ is an OVA, p is even a regular covering.

Observe that each ordinary voltage graph can be viewed as a permutation voltage graph. In fact, any element $g \in G$ determines a permutation σ_g which is given by $(x)\sigma_g = x \cdot g$, $x \in G$. The homomorphism $g \mapsto \sigma_g$ is an insertion of G into the symmetric group S_G on the set G. Let τ be a PVA given by $\tau_e = \sigma_{\varkappa(e)}$. Then $K^\varkappa = K^\tau$ and the natural projections $p : K^\varkappa \to K$ and $q : K^\tau \to K$ are equivalent, that is, there is an isomorphism (namely the identity) $j : K^\varkappa \to K^\tau$ such that $p \circ j = q$.

Let v be a vertex in K. It is well known [7] that any element α of the fundamental group $\pi(K, v)$ can be represented by an oriented closed walk in K. We identify this walk with α. It enables us to define a homomorphism $\varkappa_* : \pi(K, v) \to G$ if $\varkappa$ is an OVA and $\varkappa_* : \pi(K, v) \to S_n$ if $\varkappa$ is a PVA. Its value on $\alpha \in \pi(K, v)$ is then the product of voltages (i.e., values of $\varkappa$) on α in the order given by α. In both cases we denote the image of $\pi(K, v)$ by $\varkappa(K, v)$. In [12] it is known as the local group, but we call it the characteristic group of $\varkappa$ in v (or the monodromy group of the natural projection $p : K^\varkappa \to K$). The homomorphism $\varkappa_*$ is then called the characteristic homomorphism, and the resulting epimorphism $\pi(K, v) \to \varkappa(K, v)$, also denoted by $\varkappa_*$, the characteristic epimorphism.

Characteristic groups have been used to count the number of components of $K^{\varkappa}$ [1], [2], to determine orientability of the derived embedding [2, 6], or to detect equivalence of coverings [2]. Theorem 1, which is proved in this section, gives a characterization of the characteristic group of a voltage assignment. Such a characterization for OVA's is Theorem 3 in [1]. In our terminology it reads as follows.

Let $\varkappa$ be an OVA on K in a group G. Then $\varkappa(K, v)$ is the smallest subgroup H of G (with respect to $\subseteq$) which admits a labelling of vertices of K by (left) cosets of H in G satisfying two conditions:

(a) the vertex v is labelled by H;

(b) the labelling is consistent with $\varkappa$ (i.e., if u is labelled by A, w by B and e is an edge from u to w, then $\varkappa(e)A = B$).

This theorem is easily seen to be valid also for PVA's (with S_n instead of G).

The present characterization is of a somewhat different form and has a flavour of category theory.

Let $\varkappa$ be an OVA or a PVA on K. It is known that the covering $p: K^{\varkappa} \to K$ is completely described by the standard right action, occasionally called the monodromy of the covering, of the fundamental group $\pi(K, v)$ on the fibre $p^{-1}(v)$, (see [7]). Ezell [2] has shown that the same holds for the action of the characteristic group $\varkappa(K, v)$ on the set $p^{-1}(v)$. One can observe that these two actions are essentially the same. We prove that the characteristic group is in fact the smallest to act on the fibre the same way as $\pi(K, v)$.

To make the above statements precise, let us denote by $\mathscr{A}(p)$ the class of all (not necessarily finite) groups G satisfying the following properties:

1. G has a right transitive action on $p^{-1}(v)$;
2. there exist an epimorphism $f: \pi(K, v) \to G$ such that for any $w \in p^{-1}(v)$ and any $\alpha \in \pi(K, v)$ it holds

$$w \cdot \alpha = w \cdot f(\alpha). \tag{1}$$

Obviously, $\varkappa(K, v)$ belongs to $\mathscr{A}(p)$. Moreover, we have

Theorem 1. *For any group G in $\mathscr{A}(p)$ there is an epimorphism $g: G \to \varkappa(K, v)$ such that for an arbitrary vertex $w \in p^{-1}(v)$ and for any element $x \in G$ we have $w \cdot x = w \cdot g(x)$.*

In other words, $\varkappa(K, v)$ is a terminal object in a category with the object class $\mathscr{A}(p)$ and the action-preserving epimorphisms as morphisms.

Proof. We restrict ourselves to the PVA case since it implies also the OVA case.

Suppose $\varkappa$ is a PVA with voltages in a symmetric group S_n. Let $f: \pi(K, v) \to G$ be an epimorphism satisfying the condition (1) and $\varkappa_*: \pi(K, v) \to \varkappa(K, v)$ the characteristic epimorphism. To carry out the proof it is sufficient to show that there is a homomorphism $g: G \to \varkappa(K, v)$ which makes the diagram

$$\begin{tikzcd} \pi(K, v) \arrow[r, "f"] \arrow[dr, "\varkappa_*"'] & G \arrow[d, dashed, "g"] \\ & \varkappa(K, v) \end{tikzcd}$$

commute.

Take an arbitrary vertex v_i in $p^{-1}(v)$ and an element α in $\pi(K, v)$. Then $v_i \cdot \alpha = v_j$ (for some $j \in \{1, 2, \ldots, n\}$) iff $i(\varkappa_*(\alpha)) = j$ or, which is the same, iff $v_i \cdot \varkappa_*(\alpha) = v_j$. Since $v_i \cdot f(\alpha) = v_j$, too, it is natural to set $g: x = f(\alpha) \mapsto \varkappa_*(\alpha)$. However, it is not clear whether the definition is correct. To see it, suppose $f(\alpha) = x = f(\beta)$. Then $v_i \cdot \alpha = v_i \cdot f(\alpha) = v_i \cdot f(\beta) = v_i \cdot \beta$. Therefore $v_i \cdot \varkappa_*(\alpha) = v_i \cdot \varkappa_*(\beta)$, which is the same as $i(\varkappa_*(\alpha)) = i(\varkappa_*(\beta))$. Since $i \in \{1, 2, \ldots, n\}$ may be chosen arbitrarily, we obtain $\varkappa_*(\alpha) = \varkappa_*(\beta)$.

This completes the proof. □

2. Equivalence of coverings

As usual, two coverings $p_1: K_1 \to K$ and $p_2: K_2 \to K$ of the same graph K are said to be equivalent if there is an isomorphism $f: K_1 \to K_2$ that makes the diagram

$$\begin{tikzcd} K_1 \arrow[rr, dashed, "f"] \arrow[dr, "p_1"'] & & K_2 \arrow[dl, "p_2"] \\ & K & \end{tikzcd}$$

commute.

Ezell [2, Theorem 4.1] proved that, in the general case, two n-fold coverings $f_1: X_1 \to X$ and $f_2: X_2 \to X$ of the same topological space X are equivalent iff there is an element $\sigma \in S_n$ such that for any $\alpha \in \pi(X, x)$ it holds that

$$\sigma \circ \varphi_1(\alpha) \circ \sigma^{-1} = \varphi_2(\alpha).$$

Here φ_1 and φ_2 are the homomorphisms from $\pi(X, x)$ to S_n induced by f_1 and f_2, respectively.

If f_1 and f_2 are coverings generated by voltage graphs then the homomorphisms φ_1 and φ_2 are in fact the characteristic homomorphisms to S_n.

It would be convenient, however, to compare graph coverings directly from voltage assignments. To make it possible, we give one more definition. It is also due to Ezell [2], but it may be found in [6], as well.

Let $\varkappa$ be an OVA or a PVA on K. Choose a spanning tree T and a vertex v in K. For any edge e which is not in T there is an oriented closed walk α_e in K, based at v, with the following properties:

(a) $\alpha_e - e \subseteq T$;

(b) the orientation of α_e agrees with that of e;

(c) α_e has the minimum length among all walks satisfying (a) and (b).

Putting $\varkappa'(e) = \varkappa_*(\alpha_e)$, if e is not in T, and $\varkappa'(e) = 1$ otherwise, we obtain a new voltage assignment $\varkappa'$ called the (T, v)-reduction of $\varkappa$.

Ezell's Theorem 4.1 in [2] implies that $\varkappa$ and $\varkappa'$ generate equivalent coverings. In fact, $\varkappa_* = \varkappa'_*$.

Now, the basic idea is to compare voltage assignments by their reductions. We realize it in our Theorem 2.

Theorem 2. *Let $\varkappa$ and λ be two OVA's on the graph K, both in the same group G. Further, let $\varkappa'$ and λ' be their (T, v)-reductions. The coverings $p: K^\varkappa \to K$ and $q: K^\lambda \to K$ are equivalent iff there is an automorphism A of the group G such that $A \circ \varkappa' = \lambda'$.*

To obtain an analogue theorem for PVA, one has to replace the word "automorphism" by "inner automorphism" and, of course, "group G" by "symmetric group S_n".

Proof. We first prove the PVA version of the Theorem 2 since it is, essentially, a simple reformulation of the Theorem 4.1 of [2].

Let $\varkappa$ and λ be PVA's. As mentioned above, we have $\varkappa_* = \varkappa'_*$ and $\lambda_* = \lambda'_*$. Assume that A is an inner automorphism of S_n. The homomorphisms $A \circ \varkappa'_*$ and λ'_* are defined on $\pi(K, v)$ which is a free group. Therefore, they are equal iff they coincide on free generators. On the other hand, $A \circ \varkappa' = \lambda'$ iff $A \circ \varkappa'(e) = \lambda'(e)$ for edges e not in T. In this case, $\varkappa'(e) = \varkappa_*(\alpha_e)$, and α_e are known to form a set of free generators of $\pi(K, v)$. The proof for the PVA case follows.

Now, let $\varkappa$ and λ be OVA's. Let ψ and ω be the PVA's defined by the OVA's $\varkappa'$ and λ', respectively. According to the first part of the proof, the coverings p and q are equivalent iff there is an inner automorphism I of S_n such that $I \circ \psi = \omega$.

Suppose we have such an automorphism I. If we write G' instead of $i(G)$ (recall that i is the insertion of G into S_G) then $I(G') = G'$. The last equality follows from the way ψ and ω were defined. Therefore $I \mid G'$ is an automorphism of the group G'. This fact implies that there is an automorphism A of the group G such that $A \circ \varkappa' = \lambda'$.

Conversely, if there is an automorphism A of G such that $A \circ \varkappa' = \lambda'$, it specifies a relabelling of permuted elements of the group S_G. Thus, A induces an inner automorphism I of S_G. Since $A \circ \varkappa' = \lambda'$, we have also $I \circ \psi = \omega$. □

The natural question is, what happens if we do not insist on the condition that $\varkappa$ and λ should have the same voltage group?

If $\varkappa$ and λ are PVA's in non-isomorphic symmetric groups then p and q cannot be equivalent: they differ in the number of sheets.

In OVA's the situation is rather different. Suppose G_1 and G_2 are non-isomorphic groups of the same order. Further, let $J_1 \subseteq G_1$ and $J_2 \subseteq G_2$ be their isomorphic subgroups and $j: J_1 \to J_2$ an isomorphism. Finally, let $\varkappa$ be a (T, v)-reduction of some OVA in G_1 with the characteristic group $\varkappa(K, v)$ equal to J_1. Now, put $\lambda(e) = j \circ \varkappa(e)$ for any edge e in K. Obviously, λ is an OVA in G_2 with $\lambda(K, v) = J_2$. The number of components of a derived graph is known to be equal to the index of the characteristic group in a voltage group (see [1]). If $\tilde{K}$ is a connected component of $K^{\varkappa}$ and $\tilde{L}$ a connected component of K^{λ} then $p \mid \tilde{K}$ and $q \mid \tilde{L}$ are equivalent coverings. Since components of a derived graph are, in the OVA case, mutually isomorphic [1], choosing appropriate isomorphisms between components of $K^{\varkappa}$ and components of K^{λ} we find that the coverings p and q are equivalent.

The last fact implies that p and q have isomorphic automorphism groups. Hence G_1 and G_2 are not the automorphism groups of p and q, respectively, as one could believe. This belief comes true only when the derived graph is connected.

In the following section, which is devoted to an investigation of automorphism groups of coverings generated by voltage assignments, we determine the automorphism groups even in the case when the derived graph is disconnected.

3. Automorphism groups

Given two connected graphs $\tilde{K}$, K and a covering $p: \tilde{K} \to K$, by $\mathrm{Aut}(p)$ we denote the group of automorphisms (= self-equivalences) of p. Let $p_*: \pi(\tilde{K}, \tilde{v}) \to \pi(K, v)$ be the usual induced homomorphism of the corresponding fundamental groups. It is well known [7] that $\mathrm{Aut}(p)$ is isomorphic to the quotient group of the normalizer $N(p_*\pi(\tilde{K}, \tilde{v}))$ of $p_*\pi(\tilde{K}, \tilde{v})$ in $\pi(K, v)$ by $p_*\pi(\tilde{K}, \tilde{v})$:

$$\mathrm{Aut}(p) \cong N(p_*\pi(\tilde{K}, \tilde{v}))/p_*\pi(\tilde{K}, \tilde{v}). \tag{2}$$

The main result of this section is that, if p is generated by a voltage assignment, $p_*\pi(\tilde{K}, \tilde{v})$ can be replaced by a stabilizer of some element and $\pi(K, v)$ by the characteristic group obtaining an analogous formula (4) (see below) for $\mathrm{Aut}(p)$ as the one above in (2).

Let G be a group and S a set and suppose that G acts transitively on S. An action-preserving bijection $S \to S$ is called an automorphism of the homogeneous G-space S. The group of all automorphisms of S is denoted by $\mathrm{Aut}(G, S)$. In the following series of propositions we reduce the formula (2) to the formula (4). Although the homogeneity (or transitivity) may be a superfluous assumption, we do not consider other G-spaces.

First, let us consider the standard right action of $\pi(K, v)$ on $p^{-1}(v)$.

Proposition 1. $\mathrm{Aut}(p) \cong \mathrm{Aut}(\pi(K, v), p^{-1}(v))$.

Proof. See [7], Theorem 7.2. $\square$

If S is a (homogeneous) G-space and $s \in S$ let $\mathrm{Stab}_G\, s = \{g \in G: s \cdot g = s\}$ be a stabilizer of $s \in S$ in G. Further, for any subgroup $H \subseteq G$ define $N_G(H) = \{g \in G: gHg^{-1} = H\}$, the normalizer of H in G.

Proposition 2. *If S is a homogeneous G-space, then for any $s \in S$*

$$\mathrm{Aut}(G, S) \cong N_G(\mathrm{Stab}_G(s))/\mathrm{Stab}_G(s).$$

Proof. [7], Appendix B, Theorem 2.2. $\square$

Proposition 3. *Suppose that groups G and H act transitively on the set S and $f: G \to H$ is an action-preserving epimorphism. Then*
(i) $f(\mathrm{Stab}_G\, s) = \mathrm{Stab}_H\, s$,
(ii) $\mathrm{Aut}(G, S) = \mathrm{Aut}(H, S)$.

Proof. According to the assumption, for any $g \in G$ and $h \in H$ with $f(g) = h$, and for an arbitrary $s \in S$ we have

$$s \cdot g = s \cdot h. \tag{3}$$

Now one sees that (i) is trivial and (ii) follows directly from the definition and from (3). $\square$

In the following theorem recall that $\mathscr{A}(p)$ is the class of all groups (which may be infinite) acting on $p^{-1}(v)$ the same way as $\pi(K, v)$.

Theorem 3. *For any group G in $\mathscr{A}(p)$ and for any $w \in p^{-1}(v)$ it holds*

$$\mathrm{Aut}(p) \cong N_G(\mathrm{Stab}_G\, w)/\mathrm{Stab}_G\, w. \quad \square \tag{4}$$

Corollary 1. *Let $\varkappa$ be a PVA on the graph K in S_n such that $K^\varkappa$ is connected (i.e., $\varkappa(K, v)$ acts transitively on $\{1, 2, \ldots, n\}$, [2]). Let $p: K^\varkappa \to K$ be the natural projection and* $\mathrm{Stab}(i) = \{\sigma \in \varkappa(K, v): (i)\sigma = i\}$. *Then for any $i \in \{1, 2, \ldots, n\}$*

$$\mathrm{Aut}(p) \cong N_{\varkappa(K,v)}(\mathrm{Stab}(i))/\mathrm{Stab}(i). \quad \square$$

Corollary 2. *Let $\varkappa$ be an OVA on the graph K with voltages in a group G. If $K^\varkappa$ is connected (i.e., $\varkappa(K, v) = G$, [1]), then* $\mathrm{Aut}(p) = G$.

Now we pass to a special kind of voltage graphs known as Schreier graphs.
Given a group G, a subgroup H and a sequence X of generators of G, the

(right) Schreier coset graph $S(G/H, X)$ is the graph whose vertices are right cosets of H in G and for each $A \in G/H$ and x in X there exist a unique edge with the initial vertex A and the terminal vertex $A \cdot x$. This edge is said to carry the voltage x or to have the colour x. In the case when H is trivial, the graph $S(G/H, X)$ is usually called the Cayley (colour) graph and is denoted by $C(G, X)$. Let H and I be two subgroups of G such that $H \subseteq I$. The natural mapping $p: G/H \to G/I$ given by $A \mapsto B$ iff $A \subseteq B$, induces a covering, still denoted by p, $S(G/H, X) \to S(G/I, X)$ called the natural projection.

Remark. Observe that $S(G/H, X)$ may be non-isomorphic to the graph derived from $S(G/I, X)$ by the natural voltage assignment v on $S(G/I, X)$ given by X. To derive $S(G/H, X)$ from $S(G/I, X)$ one can use so called relative voltage construction introduced in [5] employing a voltage assignment λ in a group I with respect to H (see [5] for details). The assignment can be defined as follows. Let $G/I = \{I_1, I_2, \ldots, I_r\}$ and $g_i \in I_i$. If k is an edge running from I_i to I_j in $S(G/I, X)$, put $\lambda(k) = g_i \cdot v(k) \cdot g_j^{-1}$. Obviously, $g_i \cdot v(k) \cdot g_j^{-1} \in I$. Assume that L is the graph derived from $S(G/I, X)$ by means of λ and the relative voltage construction and let $q: L \to S(G/I, X)$ be the associated covering. Then q is equivalent to the natural projection p and, in particular, L is isomorphic to $S(G/H, X)$.

Corollary 3. *The group* $\mathrm{Aut}(p)$ *of automorphisms of the natural projection* $p: S(G/H, X) \to S(G/I, X)$ *is isomorphic to* $N_I(H)/H$.

Proof. Note that I is a vertex of $S(G/I, X)$ and $p^{-1}(I) = I/H$. Let us consider the action of the group I on the set $p^{-1}(I) = I/H$ given by the right multiplication by elements of I. It is easily seen that I belongs to $\mathscr{A}(p)$ and that $\mathrm{Stab}_I(H) = H$. The result now follows from Theorem 3. □

An interesting special case of Corollary 3 is when $I = G$. Then $S(G/I, X)$ is isomorphic to a bouquet of circles and, therefore, automorphisms of p are exactly those automorphisms of $S(G/H, X)$ which preserve orientations and colours of edges in $S(G/H, X)$. Thus we have

Corollary 4. *The group of all automorphisms of a Schreier coset graph* $S(G/H, X)$ *which preserve orientations and colours of edges is independent of* X *and is isomorphic to* $N_G(H)/H$.

If, moreover, we put $H = 1$, we obtain a well known result on the automorphism group of a Cayley colour graph. Namely, the group of all orientation and colour preserving automorphisms of $C(G, X)$ is isomorphic to G (see [13], Theorem 4.8). (This fact has been used, for instance, in the proof of the celebrated Frucht's theorem which claims that any finite abstract group is the group of all automorphisms of some graph.)

Before concluding this paragraph we return to the situation when the covering p is generated by an OVA. In Corollary 2 we restricted ourselves to OVA's which generate connected coverings. At this time we want to discuss the case when the derived graph is possibly disconnected.

Let $\varkappa$ be an OVA on K with voltages in a group G and $p: K^\varkappa \to K$ the derived covering. Alpert and Gross [1] proved in the dual form (i.e., for current graphs) that connected components of $K^\varkappa$ are mutually isomorphic, their number is $[G: \varkappa(K, v)]$, the index of $\varkappa(K, v)$ in G, and for any restriction q of p to a connected component of $K^\varkappa$ $\mathrm{Aut}(q) \cong \varkappa(K, v)$. The first of these assertions allows some refining. Suppose that $K^\varkappa$ consists of k connected components $L_1, L_2, \ldots, L_k$, $k = [G: \varkappa(K, v)]$, and L_1 is the component containing the vertex $(v, 1)$. Let p_i be the restriction of p to L_i. Then any two such restrictions p_i and p_j are equivalent coverings, that is, there exist isomorphisms $f_i: L_1 \to L_i$ which make the diagram

$$\begin{tikzcd} L_1 \arrow[rr, dashed, "f_i"] \arrow[dr, "p_1"'] & & L_i \arrow[dl, "p_i"] \\ & K & \end{tikzcd}$$

commute.

Our aim is to prove that the automorphism group $\mathrm{Aut}(p)$ is isomorphic to the composition (known also as the wreath product) $S_k[\varkappa(K, v)]$ of the symmetric group S_k with the group $\varkappa(K, v)$. We recall the definition of the composition $A[B]$ of two permutation groups A and B on sets $X = \{x_1, x_2, \ldots, x_d\}$ and $Y = \{y_1, y_2, \ldots, y_e\}$, respectively. The group $A[B]$ is the permutation group on the set $X \times Y$. For any $\alpha \in A$ and a sequence $\beta_1, \beta_2, \ldots, \beta_d$ of $d = |X|$ permutations which belong to B there is a permutation in $A[B]$ denoted by $(\alpha; \beta_1, \ldots, \beta_d)$ such that for $(x_i, y_j) \in X \times Y$ it holds $(x_i, y_j)(\alpha; \beta_1, \ldots, \beta_d) = ((x_i)\alpha, (y_j)\beta_i)$. In this context $\varkappa(K, v)$ is looked at as a permutation group on itself and any element $g \in \varkappa(K, v)$ is identified with the corresponding automorphism of the covering $p_1 = p \mid L_1$.

Consider again the graph $K^\varkappa$. First of all note that $p^{-1}(v) = \{(v, g): g \in \varkappa(K, v)\}$. To obtain a notational advantage, we identify $f_i(x, g)$, where x is a vertex or an edge of K and $g \in \varkappa(K, v)$, with the pair $(i, (x, g))$. Now we shall construct an isomorphism $\Phi: \mathrm{Aut}(p) \to S_k[\varkappa(K, v)]$, $\varphi \mapsto (\alpha^\varphi; g_1^\varphi, \ldots, g_k^\varphi)$ with $\alpha^\varphi \in S_k$ and $g_1^\varphi, \ldots, g_k^\varphi \in \varkappa(K, v)$. If $\varphi \in \mathrm{Aut}(p)$, then, since φ sends components to components, $\varphi(L_i)$ is equal to some L_j. Hence we can set $(i)\alpha^\varphi = j$ iff $\varphi(L_i) = L_j$. Assume that $(i)\alpha^\varphi = j$. Since $(v, 1)$ is in L_1, $f_j^{-1} \circ \varphi \circ f_i(v, 1)$ is again in L_1, and therefore $f_j^{-1} \circ \varphi \circ f_i(v, 1) = (v, g)$ for some $g \in \varkappa(K, v)$. Set $g_i^\varphi = g$. Routine calculations show that Φ is a homomorphism. It is readily verified, too, that Φ is injective. To check the surjectivity of Φ assume we are given $\sigma = (\alpha; g_1, \ldots, g_k) \in S_k[\varkappa(K, v)]$. If x is a vertex or an edge of K and $g \in \varkappa(K, v)$ let us define φ by $\varphi(i, (x, g)) = ((i)\alpha, g \cdot g_i))$. Since any g_i corresponds to an automorphism of p_1, φ is an automorphism of p. The image $\Phi(\varphi)$ is easily seen to be equal to σ.

Thus we have proved the following theorem.

Theorem 4. *Suppose that $\varkappa$ is an OVA on K with voltages in a group G and $p:K^{\varkappa}\to K$ is the natural projection. If the index $[G:\varkappa(K,v)]$ equals to k, then the group* Aut(p) *is isomorphic to the composition $S_k[\varkappa(K,v)]$ of the symmetric group S_k, with $\varkappa(K,v)$.* □

4. Regularity of coverings

In this section we aim to prove necessary and sufficient conditions of regularity of coverings, particularly those generated by PVA's or represented by natural projections of Schreier coset graphs.

A covering $p:\tilde{K}\to K$ is defined to be regular if

$$p_*\pi(\tilde{K},\tilde{v})\trianglelefteq\pi(K,v). \tag{5}$$

(We write $H\trianglelefteq G$ if H is a normal subgroup of G.) Here and throughout this section graphs are assumed to be connected.

The following theorem translates the condition (5) to the language of the class $\mathcal{A}(p)$ and stabilizers.

Theorem 5. *Let $p:\tilde{K}\to K$ be a covering, $w\in p^{-1}(v)$ and $G\in\mathcal{A}(p)$. Then p is a regular covering if and only if* $\mathrm{Stab}_G(w)\trianglelefteq G$.

Proof. Suppose that p is regular. We then have $p_*\pi(\tilde{K},w)\trianglelefteq\pi(K,v)$. Since G belongs to $\mathcal{A}(p)$, there is an action-preserving epimorphism $f:\pi(K,v)\to G$. It is known [7] that for the standard action of $\pi(K,v)$ on $p^{-1}(v)$ it holds $\mathrm{Stab}_{\pi(K,v)}(w)=p_*\pi(\tilde{K},w)$. By Proposition 3(i), $fp_*\pi(\tilde{K},w)=\mathrm{Stab}_G(w)$. But f is onto G, therefore normal subgroups are mapped onto normal subgroups. Hence $\mathrm{Stab}_G w\trianglelefteq G$.

Conversely, let $\mathrm{Stab}_G(w)\trianglelefteq G$. Take $\alpha\in p_*\pi(\tilde{K},w)$ and $\theta\in\pi(K,v)$. By the assumption, $f(\theta\cdot\alpha\cdot\theta^{-1})=f(\theta)f(\alpha)f(\theta)^{-1}\in\mathrm{Stab}_G(w)$. The epimorphism f is action preserving, therefore $w=w\cdot f(\theta\cdot\alpha\cdot\theta^{-1})=w\cdot\theta\alpha\theta^{-1}$ which in turn implies that $\theta\cdot\alpha\cdot\theta^{-1}\in p_*\pi(\tilde{K},w)$. □

We draw two corollaries from the preceding theorem. The first one concerns permutation voltage graphs.

Corollary 5. *Suppose that $\varkappa$ is a PVA on K in S_n such that $K^{\varkappa}$ is connected. Then the following statements are equivalent:*

(i) *The natural projection $p:K^{\varkappa}\to K$ is regular;*

(ii) $\mathrm{Stab}(i) \trianglelefteq \varkappa(K, v)$ *for some* $i \in \{1, 2, \ldots, n\}$;
(iii) $\mathrm{Stab}(i) = \mathrm{Stab}(1)$ *for every* $i \in \{1, 2, \ldots, n\}$;
(iv) $\mathrm{Stab}(i)$ *is trivial for some* $i \in \{1, 2, \ldots, n\}$.

Proof. (i)$\Leftrightarrow$(ii) follows directly from Theorem 5 since $\varkappa(K, v) \in \mathscr{A}(p)$.

(ii)$\Leftrightarrow$(iii): We assumed $K^{\varkappa}$ to be connected. Thus, $\varkappa(K, v)$ is a transitive permutation group on $\{1, 2, \ldots, n\}$, [2, Theorem 5.1]. In this case stabilizers $\mathrm{Stab}(i)$ are conjugated subgroups, $i = 1, 2, \ldots, n$.

Clearly, (iv) implies (ii). To prove the converse, assume the contrary. Now we have $1 \neq \mathrm{Stab}(i) \trianglelefteq \varkappa(K, v)$. Take the quotient group $G = \varkappa(K, v)/\mathrm{Stab}(i)$. Its elements are of the form $(ji) \cdot \mathrm{Stab}(i)$, where (ji) is the transposition of i and j. We can define an action of G on $p^{-1}(v)$ by $v_k \cdot (ji)\mathrm{Stab}(i) = v_k \cdot (ji)\sigma$ for some $\sigma \in \mathrm{Stab}(i)$. This definition is independent of the choice of σ, since for any σ and τ in $\mathrm{Stab}(i)$ one finds $(ji)\sigma \cdot \tau^{-1}(ji) \in \mathrm{Stab}(i)$. Note that $\mathrm{Stab}(i) = \mathrm{Stab}(k)$, as we assume (ii). Therefore $v_k \cdot (ji)\sigma = v_k \cdot (ji)\tau$. Finally we see that G belongs to $\mathscr{A}(p)$. By Theorem 1, this is a contradiction. □

A proof of Corollary 5 without Theorem 1 can be found in [11].

Gross [4] proved that any connected regular graph of even degree is isomorphic to a Schreier coset graph $S(G/H, X)$ for a suitable group G, subgroup H and a generating sequence X. In [10] it is shown that even any covering $c: K \to L$ onto a regular graph of even degree can be realized by the natural projection p of the corresponding Schreier coset graphs. More precisely, there is a group G, subgroups $I \subseteq J \subseteq G$, generators X and isomorphisms $i: K \to S(G/I, X)$, $j: L \to S(G/J, X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} K & \xrightarrow{i} & S(G/I, X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle p} \\ L & \xrightarrow[j]{} & S(G/J, X) \end{array}$$

It is obvious that c and p are either both regular or both irregular coverings. The following corollary shows that the expected condition of regularity of p (and, at the same time, of c) is really true.

Corollary 6. *The natural projection* $p: S(G/I, X) \to S(G/J, X)$ (*with* $I \subseteq J \subseteq G$) *is a regular covering iff* $I \trianglelefteq J$.

Proof. We already know that $p^{-1}(J) = J/I$, $\mathrm{Stab}_J(I) = I$ and that $J \in \mathscr{A}(p)$. These facts and Theorem 5 immediately imply the result. □

Let us conclude the section with a remark on disconnected coverings. If $p: \tilde{X} \to X$ is a covering onto a space X with $\tilde{X}$ disconnected, one could define such a covering to be regular if the restriction of p to any (path-) connected component of $\tilde{X}$ is a regular covering. However, this definition may quite often contradict the intuitive notion of regularity: the covering is regular iff the lifts of any closed path (i.e., having identical endpoints) in X are either all closed or all open paths (that is, having different endpoints).

As an example take a bouquet B_2 of two circles c and d and the PVA in S_4 given by $c \mapsto (12)(3)(4)$, $d \mapsto (1)(2)(34)$. The derived graph has two components. Each of them is a 2-fold, and therefore regular, covering space over B_2. However, the lifting condition, as easily seen from the PVA, is false.

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